Schutz

35, $g_{\alpha\beta} = \begin{pmatrix} -e^{2\Phi(r)} & & & \\ & e^{2\Lambda(r)} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix}$, $g^{\alpha\beta} = \begin{pmatrix} -e^{-2\Phi(r)} & & & \\ & e^{-2\Lambda(r)} & & \\ & & r^{-2} & \\ & & & r^{-2}\sin^{-2}\theta \end{pmatrix}$.

(t, r, θ, ϕ).

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}).$$

$$\Rightarrow \Gamma^{t}_{\mu\nu} = \frac{1}{2}(-e^{-2\Phi(r)})(g_{t\mu,\nu} + g_{t\nu,\mu} - g_{\mu\nu,t}).$$

$$\Gamma^{r}_{\mu\nu} = \frac{1}{2}(e^{-2\Lambda(r)})(g_{r\mu,\nu} + g_{r\nu,\mu} - g_{\mu\nu,r}).$$

$$\Gamma^{\theta}_{\mu\nu} = \frac{1}{2}(r^{-2})(g_{\theta\mu,\nu} + g_{\theta\nu,\mu} - g_{\mu\nu,\theta})$$

$$\Gamma^{\phi}_{\mu\nu} = \frac{1}{2}(r^{-2}\sin^{-2}\theta)(g_{\phi\mu,\nu} + g_{\phi\nu,\mu} - g_{\mu\nu,\phi}).$$

Nontrivial values for $\Gamma^{t}_{\mu\nu}$: $\Gamma^{t}_{tt} = \frac{1}{2}(-e^{-2\Phi(r)})(g_{tt,t} + g_{tt,t} - g_{tt,t})$

$$= \frac{1}{2}(-e^{-2\Phi(r)})\, g_{tt,t} = 0.$$

$$\Gamma^{t}_{tr} = \frac{1}{2}(-e^{-2\Phi(r)})(g_{tt,r} + \cancel{g_{tr,t} - g_{tr,t}}) = \frac{1}{2}(-e^{-2\Phi(r)})\, g_{tt,r}.$$

$$\Gamma^{t}_{t\theta} = \frac{1}{2}(-e^{-2\Phi(r)})(g_{tt,\theta} + \cancel{g_{t\theta,t} - g_{t\theta,t}}) = 0.$$

$$\Gamma^{t}_{t\phi} = \frac{1}{2}(-e^{-2\Phi(r)})(g_{tt,\phi} + \cancel{g_{t\phi,t} - g_{t\phi,t}}) = 0.$$

$$\Gamma^t_{rr} = \frac{1}{2}\left(-e^{-2\Phi(r)}\right)\left(\cancel{g_{tr,r}} + \cancel{g_{tr,r}} - g_{rr,t}\right)$$

$$= \frac{1}{2}\left(-e^{-2\Phi(r)}\right)\left(-g_{rr,t}\right).$$

$$= \frac{1}{2} e^{-2\Phi(r)} g_{rr,t} = 0.$$

$$\Gamma^t_{r\theta} = \frac{1}{2}\left(-e^{-2\Phi(r)}\right)\left(g_{tr,\theta} + g_{t\theta,r} - g_{r\theta,t}\right) = 0.$$

$$\Gamma^t_{r\phi} = 0.$$

$$\Gamma^t_{\theta\theta} = \frac{1}{2}\left(-e^{-2\Phi(r)}\right)\left(g_{t\theta,\theta} + g_{t\theta,\theta} - g_{\theta\theta,t}\right) = 0.$$

$$\Gamma^t_{\theta\phi} = \frac{1}{2}\left(-e^{-2\Phi(r)}\right)\left(g_{t\theta,\phi} + g_{t\phi,\theta} - g_{\theta\phi,t}\right) = 0.$$

$$\Gamma^t_{\phi\phi} = \frac{1}{2}\left(-e^{-2\Phi(r)}\right)\left(g_{t\phi,\phi} + g_{t\phi,\phi} - g_{\phi\phi,t}\right) = 0.$$

$$\Gamma^r_{tt} = \frac{1}{2}\left(e^{-2\Lambda(r)}\right)\left(\cancel{g_{rt,t}} + \cancel{g_{rt,t}} - g_{tt,r}\right)$$

$$= \frac{1}{2}\left(e^{-2\Lambda(r)}\right)\left(-g_{tt,r}\right).$$

$$\Gamma^r_{tr} = \frac{1}{2}\left(e^{-2\Lambda(r)}\right)\left(g_{rt,r} + g_{rr,t} - g_{tr,r}\right) = 0.$$

$$\Gamma^r_{t\theta} = \frac{1}{2}\left(e^{-2\Lambda(r)}\right)\left(g_{rt,\theta} + g_{r\theta,t} - g_{t\theta,r}\right) = 0.$$

$$\Gamma^r_{t\phi} = \frac{1}{2}\left(e^{-2\Lambda(r)}\right)\left(g_{rt,\phi} + g_{r\phi,t} - g_{t\phi,r}\right) = 0.$$

$$\Gamma^r_{rr} = \frac{1}{2}\left(e^{-2\Lambda(r)}\right)\left(g_{rr,r} + g_{rr,r} - g_{rr,r}\right)$$

$$= \frac{1}{2}\left(e^{-2\Lambda(r)}\right) g_{rr,r}.$$

$$\Gamma^r_{r\theta} = \frac{1}{2}\left(e^{-2\Lambda(r)}\right)\left(g_{rr,\theta}\right) = 0.$$

$$\Gamma^r_{r\phi} = \frac{1}{2}\left(e^{-2\Lambda(r)}\right)\left(g_{rr,\phi}\right) = 0.$$

$$\Gamma^r_{\theta\theta} = \frac{1}{2}\left(e^{-2\Lambda(r)}\right)\left(-g_{\theta\theta,r}\right) = \frac{1}{2}\left(e^{-2\Lambda(r)}\right)\left(-g_{\theta\theta,r}\right).$$

$$\Gamma^r_{\theta\phi} = 0.$$

$$\Gamma^r_{\phi\phi} = \frac{1}{2}\left(e^{-2\Lambda(r)}\right)\left(-g_{\phi\phi,r}\right)$$

$$\Gamma^\theta_{tt} = \frac{1}{2}\left(r^{-2}\right)\left(g_{\theta t,t} + g_{\theta t,t} - g_{\theta\theta,t}\right)$$

$$= \frac{1}{2}\left(r^{-2}\right)\left(-g_{\theta\theta,t}\right) = 0.$$

$$\cancel{\Gamma^\theta_{\theta\theta}}\ \Gamma^\theta_{tr} = 0, \quad \Gamma^\theta_{t\theta} = 0, \quad \Gamma^\theta_{t\phi} = 0.$$

$$\Gamma^\theta_{rr} = \frac{1}{2}\left(r^{-2}\right)\left(-g_{rr,\theta}\right) = 0.$$

$$\Gamma^\theta_{r\theta} = \frac{1}{2}\left(r^{-2}\right)\left(g_{\theta r,\theta} + g_{\theta\theta,r} - g_{r\theta,\theta}\right)$$

$$= \frac{1}{2}\left(r^{-2}\right)\left(g_{\theta\theta,r}\right).$$

$$\Gamma^\theta_{r\phi} = 0,$$

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2}\left(r^{-2}\right)\left(g_{\theta\theta,\theta} + \cancel{g_{\theta\theta,\theta} - g_{\theta\theta,\theta}}\right)$$

$$= \frac{1}{2}\left(r^{-2}\right)\left(g_{\theta\theta,\theta}\right) = 0.$$

$$\Gamma^{\theta}_{t\phi} = \frac{1}{2}(r^{-2})(g_{\theta\theta,\phi}) = 0.$$

$$\Gamma^{\theta}_{\phi\phi} = \frac{1}{2}(r^{-2})(-g_{\phi\phi,\theta}).$$

$$\Gamma^{\phi}_{tt} = \frac{1}{2}(r^{-2}\sin^{-2}\theta)(-g_{tt,\phi}) = 0.$$

$$\Gamma^{\phi}_{tr} = 0, \quad \Gamma^{\phi}_{t\theta} = 0.$$

$$\Gamma^{\phi}_{t\phi} = \frac{1}{2}(r^{-2}\sin^{-2}\theta)(g_{\phi\phi,t}) = 0.$$

$$\Gamma^{\phi}_{rr} = \frac{1}{2}(r^{-2}\sin^{-2}\theta)(-g_{rr,\phi}) = 0.$$

$$\Gamma^{\phi}_{r\theta} = 0, \quad \Gamma^{\phi}_{r\phi} = \frac{1}{2}(r^{-2}\sin^{-2}\theta)(g_{\phi\phi,r}).$$

$$\Gamma^{\phi}_{\theta\theta} = \frac{1}{2}(r^{-2}\sin^{-2}\theta)(-g_{\theta\theta,\phi}) = 0.$$

$$\Gamma^{\phi}_{\theta\phi} = \frac{1}{2}(r^{-2}\sin^{-2}\theta)(g_{\phi\phi,\theta}).$$

$$\Gamma^{\phi}_{\phi\phi} = \frac{1}{2}(r^{-2}\sin^{-2}\theta)(g_{\phi\phi,\phi} + g_{\phi\phi,\phi} - g_{\phi\phi,\phi})$$

$$= \frac{1}{2}(r^{-2}\sin^{-2}\theta)(g_{\phi\phi,\phi}) = 0.$$

We have found all nonvanishing Γs:

$$\Gamma^{t}_{tr} = \frac{1}{2}\left(-e^{-2\Phi(r)}\right) g_{tt,r} = \frac{d\Phi(r)}{dr}$$

$$\Gamma^{r}_{tt} = \frac{1}{2}\left(e^{-2\Lambda(r)}\right)\left(-g_{tt,r}\right) = \frac{d\Phi}{dr} e^{2[\Phi-\Lambda]}$$

$$\Gamma^{r}_{rr} = \frac{1}{2}\left(e^{-2\Lambda(r)}\right) g_{rr,r} = \frac{d\Lambda(r)}{dr}$$

$$\Gamma^{r}_{\theta\theta} = \frac{1}{2}\left(e^{-2\Lambda(r)}\right)\left(-g_{\theta\theta,r}\right) = -r e^{-2\Lambda(r)}$$

$$\Gamma^{r}_{\phi\phi} = \frac{1}{2}\left(e^{-2\Lambda(r)}\right)\left(-g_{\phi\phi,r}\right) = -r \sin^2\theta\, e^{-2\Lambda(r)}$$

$$\Gamma^{\theta}_{r\theta} = \frac{1}{2}\left(r^{-2}\right) g_{\theta\theta,r} = r^{-1}$$

$$\Gamma^{\theta}_{\phi\phi} = \frac{1}{2}\left(r^{-2}\right)\left(-g_{\phi\phi,\theta}\right) = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}_{\theta\phi} = \frac{1}{2}\left(r^{-2}\sin^{-2}\theta\right)\left(g_{\phi\phi,\theta}\right) = \sin^{-1}\theta\cos\theta$$

We identify the independent components of R by $\alpha\beta\mu\nu$, by symmetry, we can map the independent components on the following grid.

	tr	$t\theta$	$t\phi$	$r\theta$	$r\phi$	$\theta\phi$
tr	•	•	•	•	•	•
$t\theta$	•	•	•	•	•	
$t\phi$	•	•	•	•		
$r\theta$	•	•	•			
$r\phi$	•	•				
$\theta\phi$	•					

$$R^t{}_{rtr} = \Gamma^t_{rr,t} - \Gamma^t_{rt,r} + \Gamma^t_{\sigma t}\Gamma^\sigma_{rr} - \cancel{\Gamma^t_{rt}}\,\Gamma^t_{\sigma r}\Gamma^\sigma_{rt}$$

$$= -\frac{d^2\Phi(r)}{dr^2} + \Gamma^t_{rt}\Gamma^r_{rr} - \Gamma^t_{tr}\Gamma^t_{rt}$$

$$= \boxed{-\frac{d^2\Phi(r)}{dr^2} + \frac{d\Phi}{dr}\frac{d\Lambda}{dr} - \frac{d\Phi}{dr}\frac{d\Phi}{dr}}$$

$$R^t{}_{rt\theta} = \Gamma^t_{r\theta,t} - \Gamma^t_{rt,\theta} + \Gamma^t_{\sigma t}\Gamma^\sigma_{r\theta} - \Gamma^t_{\sigma\theta}\Gamma^\sigma_{rt}$$

$$= 0 - 0 + \Gamma^t_{rt}\Gamma^r_{r\theta} - \cancel{\Gamma^t}$$

$$= \frac{d\Phi}{dr}[0] = \boxed{0.}$$

$$R^t{}_{rt\phi} = \Gamma^t_{r\phi,t} - \Gamma^t_{rt,\phi} + \Gamma^t_{\sigma t}\Gamma^\sigma_{r\phi} - \Gamma^t_{\sigma\phi}\Gamma^\sigma_{rt}$$

$$= 0 - 0 + \Gamma^t_{rt}\Gamma^r_{r\phi} - \cancel{\Gamma^t}\ 0 = \boxed{0.}$$

$$R^t{}_{rr\theta} = \Gamma^t_{r\theta,r} - \Gamma^t_{rr,\theta} + \Gamma^t_{\sigma r}\Gamma^\sigma_{r\theta} - \Gamma^t_{\sigma\theta}\Gamma^\sigma_{rr}$$

$$= 0 + \Gamma^t_{tr}\Gamma^t_{r\theta} - \cancel{\Gamma}\ 0 = \boxed{0.}$$

$$R^t{}_{rr\phi} = \Gamma^t_{r\phi,r} - \Gamma^t_{rr,\phi} + \Gamma^t_{\sigma r}\Gamma^\sigma_{r\phi} - \Gamma^t_{\sigma\phi}\Gamma^\sigma_{rr}$$

$$= 0 + 0 = \boxed{0}$$